UGEB2530 Game and strategic thinking
Solution to Assignment 3

Due: No need to submit

1. Consider the subtraction game with substraction set $S = \{1, 2, 4\}$.
   (a) Draw the tree diagram for the game if initially there are 5 chips.
   (b) Use backward induction to determine whether the first or the second player has a winning strategy if initially there are 5 chips.
   (c) Determine whether n is a P-position or an N-position for $n = 9, 10, 11, 12$.
   (d) Determine whether n is a P-position or an N-position for $n = 100, 101, 102, 103$.

   Solution:

   (a) Omitted.
   (b) The second player has a winning strategy.
   (c)

0	1	2	3	4	5	6	7	8	9	10	11	12
P	N	N	P	N	N	P	N	N	P	N	N	P

   (d) Position n is a P-position if n is divided by 3. So we have:

100	101	102	103
N	N	P	N

2. In a 2-pile take-away game, there are 2 piles of chips. In each turn, a player may either remove any number of chips from one of the piles, or remove the same number of chips from both piles. The player removing the last chip wins.
   (a) Find all winning moves for the starting positions $(6, 9)$, $(11, 15)$ and $(13, 20)$.
   (b) Find (x, y) if (x, y) is a P-position and
      (i) $x = 70$
      (ii) $x = 100$
      (iii) $x - y = 200$

   Solution:

   (a) $(6, 9)$ is a N-position, using the PN-diagram, there is only one winning move that is $(6, 9) \to (4, 7)$ which is a P-position.
   $(11, 15)$ is a N-position, using the PN-diagram, there are one winning moves: $(11, 15) \to (6, 10)$, $(11, 15) \to (9, 15)$, which move to a P-position.
   $(13, 20)$ is a N-position, using the PN-diagram, there are winning moves: $(13, 20) \to (12, 20)$, $(13, 20) \to (11, 18)$ which move to a P-position.

   (b) i. To find n such that $[n\varphi] = 70$ or $[n\varphi] + n = 70$, that is $70 \leq n\varphi < 71$ or $70 \leq n(\varphi + 1) < 71$.
   Find that $n = 27$ satisfies $70 \leq n(\varphi + 1) < 71$, so $(x, y) = (70, 43)$.

ii. To find n such that $[n\varphi] = 100$ or $[n\varphi] + n = 100$, that is $100 \leq n\varphi < 101$ or $100 \leq n(\varphi + 1) < 101$.
Find that $n = 62$ satisfies $100 \leq n\varphi < 101$, so $(x, y) = (100, 162)$.

iii. $(x, y) = ([200\varphi] + 200, [200\varphi]) = (523, 323)$.

3. Find x, where $\oplus$ denotes the nim-sum.

   (a) $x = 3 \oplus 6$

   (b) $x = 13 \oplus 22 \oplus 25$

   (c) $x \oplus 13 = 20$

   Solution:

   (a) For $3 = 11_2$, $6 = 111_2$. So $x = 100_2 = 4$.

   (b) For $13 = 1101_2$, $22 = 10110_2$, $25 = 11001_2$. So $x = 10_2 = 2$.

   (c) For $x = x \oplus 13 \oplus 13 = 20 \oplus 13$, and $13 = 1101_2$, $20 = 10100_2$. So $x = 11001_2 = 25$.

4. Determine whether the following positions are P-position or N-position in the game of nim. If it is an N-position, determine all winning strategies for the next player.

   (a) $(5, 9, 12)$

   (b) $(5, 11, 12)$

Solution:

1. Write the numbers in binary form $5 = 101_2, 9 = 1001_2, 12 = 1100_2$.
   Consider the nim-sum $5 \oplus 9 \oplus 12 = 0$ So it is a P-position.

2. Write the numbers in binary form $5 = 101_2, 11 = 1011_2, 12 = 1100_2$.
   Consider the nim-sum $5 \oplus 9 \oplus 12 = 0$ So it is a N-position. It has one winning move to $(5, 9, 12)$.